

ON MONOMIAL REPRESENTATIONS OF FINITELY GENERATED NILPOTENT GROUPS

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ABSTRACT. A result of D. Segal states that every complex irreducible representation of a finitely generated nilpotent group G is monomial if and only if G is abelian-by-finite. A conjecture of A. N. Parshin, recently proved affirmatively by I.V. Beloshapka and S. O. Gorchinskii (2016), characterizes the monomial irreducible representations of finitely generated nilpotent groups. This article gives a slightly shorter proof of the conjecture combining the ideas of I. D. Brown and P. C. Kutzko. We also characterize finite dimensional irreducible representations of two step nilpotent groups and also provide a full description of the finite dimensional representations of two step groups whose center has rank one.

1. INTRODUCTION

It is well known that every finite-dimensional irreducible representation of a nilpotent group over the field of complex numbers is monomial, that is induced from a one dimensional representation of some subgroup. Hall [10], proved that every complex irreducible representation of a finitely generated nilpotent group G is finite dimensional if and only if G is abelian-by-finite, i.e., G contains an abelian normal subgroup N such that G/N is finite. Therefore, in general a finitely generated nilpotent group has infinite-dimensional irreducible representations. As a first step towards an understanding of representations of finitely generated nilpotent groups, one would like to know whether all irreducible representations are necessarily monomial. Segal [17], proved that every complex irreducible representation of G is monomial if and only if G is abelian-by-finite. Therefore the question of characterizing monomial irreducible representations of a finitely generated nilpotent groups arises naturally. The characterization that works efficiently comes from the parallel questions for the unitary irreducible representations of nilpotent Lie groups. For the motivation, we recall these results first.

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For the case of simply connected nilpotent Lie groups and complex unitary irreducible representations the analogous result appeared in 1962 in the classical work of Kirillov [11] (see also Dixmier [7, 8]), where he developed the famous orbit method to construct irreducible representations of simply connected nilpotent Lie groups. It turns out that every unitary irreducible representation of a simply connected nilpotent Lie group is induced from a unitary character of a subgroup.

On the other hand, the complex irreducible unitary representations of finitely generated discrete nilpotent groups are not necessarily monomial. For example, I.D. Brown [5] constructed an irreducible unitary representation of discrete Heisenberg group that is not monomial. Brown [5] also proved that a unitary irreducible representations of a discrete nilpotent group is monomial if and only if it has finite weight. Recall that a representation ρ of G is said to have finite weight if there exists a subgroup H of G and a character χ of H such that space of H -linear maps, also called the space of intertwining operators, $\text{Hom}_H(\rho|_H, \chi)$ is non-zero and finite dimensional.

Brown's result [5] motivates the question whether a similar characterization holds for irreducible algebraic representations of finitely generated discrete nilpotent groups. That is, if G is such a group and π is an irreducible algebraic representation of G , then is it true that π is equivalent to $\text{Ind}_H^G(\chi)$ for some character χ of a subgroup H if and only if π has finite weight. A. N. Parshin [15], during plenary lecture at ICM 2010, conjectured that Brown's characterization of monomial representations holds for all complex irreducible representations of finitely generated nilpotent groups (see also [1]). In this case by induction of a representation, we mean finite induction (See Definition 2.1). This conjecture was proved by Beloshapka and Gorchinskii (see [4]).

In this article our aim is to present a proof of this conjecture following the ideas of Brown [5]. An earlier version of this article (which was circulated among some) which claimed to have settled Parshin's conjecture contained a gap in the proof. This was pointed out to us by Prof. Parshin (see Proposition 2.8). Beloshapka and Gorchinskii, working on the same problem independently, fixed this gap, thus proving the conjecture in full (see [4]). Although, both these proofs were modeled on the proof by Brown [5] for unitary representations, we feel that our usage of Kutzko's results (see Theorem 2.3) makes the proof of the main theorem shorter. The results of Sections 4 and 5, we believe are new and are of independent interest. The following is the main result of this paper. We build on the ideas of this article to prove parallel results for finitely generated supersolvable groups in [14]. Unexpectedly, the proofs get much more involved for the supersolvable case as compared to the nilpotent one. It will be interesting to characterize the class of infinite polycyclic groups for which Brown's characterization holds.

Theorem 1.1. *Let G be a finitely generated nilpotent group. An irreducible countable dimensional representation π of G is monomial if and only if it has finite weight.*

We describe the strategy of the proof here. First of all we obtain an important sufficient condition for a subgroup H of a nilpotent group G and its character χ such that $\text{End}_G(\text{Ind}_H^G(\chi)) \cong \mathbb{C}$. For this we use a result of Kutzko [12] and a few ideas of Brown [5]. Then we show that there exists a subgroup H' and its character χ' such that $\text{End}_G(\text{Ind}_{H'}^G(\chi')) \cong \mathbb{C}$ and $V_{H'}(\chi')$ is non-trivial. In the proof we see that finite weight condition is used crucially. Next, we establish a non-trivial G -linear map between π (given finite weight representation) and $\text{Ind}_{H'}^G(\chi')$. Then we indicate the proof of the fact that $\text{Ind}_{H'}^G(\chi')$ is irreducible for above obtained H' and χ' . Combining this all together, we obtain $\pi \cong \text{Ind}_{H'}^G(\chi')$ and therefore π is monomial.

Our next results are regarding the description of finite dimensional representations of the finitely generated two step nilpotent groups. First of all, we provide a full description of the finite dimensional representations of two step groups whose center has rank one.

Define $\mathbb{H}(s_1, s_2, \dots, s_n) = \{(u, v, z) \mid z \in \mathbb{Z}, u, v \in \mathbb{Z}^n\}$ where the group operation is defined by,

$$(u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n, z)(u'_1, u'_2, \dots, u'_n, v'_1, v'_2, \dots, v'_n, z') = \\ (u_1 + u'_1, \dots, v_n + v'_n, z + z' + \sum_{i=1}^n s_i u_i v'_i)$$

It is well known that a two step nilpotent group with rank one center is of the form $\mathbb{H}(s_1, s_2, \dots, s_n)$.

Theorem 1.2. *Let $G = \mathbb{H}(s_1, s_2, \dots, s_n)$ and (ρ, V) be an irreducible monomial representation of G with respect to $(H(V), \chi)$. Then the following are equivalent.*

- (1) $\chi|_{[G, G]}$ is of order $N_1 < \infty$.
- (2) V is finite dimensional. Further for all $1 \leq j \leq n$, let N_j be the least positive integer such that $\frac{N_j s_j}{s_1}$ is a multiple of N_1 . Then $\dim(V) = N_1 N_2 \dots N_n$.

This is a result analogous to Theorem 4 in [2].

In the end, we look at finite dimensional representations of finitely generated two step nilpotent groups more closely and prove the following.

Theorem 1.3. *Let G be a finitely generated two step nilpotent group and ρ be an irreducible representation of G . Then ρ is finite dimensional if and only if character χ obtained by restricting ρ to $[G, G]$ has finite order. Further, in this case the following hold.*

- (1) *order of χ divides $\dim(\rho)$.*
- (2) *$\dim(\rho) = \sqrt{|G_\rho/Z(G_\rho)|}$ where $G_\rho = G/\text{Ker}(\rho)$.*

2. PRELIMINARIES

In this section we recall basic definitions, fix notation and prove few important results that are required to prove Theorem 1.1.

Let G be a group. An algebraic representation of G is a homomorphism, $\pi : G \rightarrow \text{Aut}(V)$, from group G to the set of automorphisms of a complex vector space V . We remark that V can possibly be infinite dimensional. We denote this by (π, V) . We also denote this just by either π or V whenever our meaning is clear from the context. If V is a finite dimensional vector space, we say π is a finite dimensional representation of G and its dimension is equal to that of V . The induced representation is defined as follows.

Definition 2.1. (Induced representation) Let H be a subgroup of a group G and (ρ, W) be a representation of H . The induced representation $(\tilde{\rho}, \tilde{W})$ of ρ from H to G has representation space \tilde{W} consisting of functions $f : G \rightarrow \text{End}(W)$ satisfying the following:

- (1) $f(hg) = \rho(h)f(g)$ for all $g \in G, h \in H$.
- (2) Support of f is contained in a set of finite number of right cosets of H in G .

Then the homomorphism $\tilde{\rho} : G \rightarrow \text{Aut}(\tilde{W})$ is given by $\tilde{\rho}(g)f(x) = f(xg)$ for all $x, g \in G$. We denote this induced representation by $\text{Ind}_H^G(\rho)$.

In the language of group algebras, induced representation \tilde{W} of G , satisfies $\tilde{W} \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ with the action of G given by multiplication on the left. We remark that such representations have been already studied in literature (see for example [16]).

Definition 2.2. (Finite weight representation) A representation (π, V) of a group G is said to have **finite weight** if there is a subgroup H and a character $\chi : H \rightarrow \mathbb{C}^\times$ such that the space $V_H(\chi)$ defined by,

$$V_H(\chi) = \{v \in V \mid \pi(h)v = \chi(h)v \text{ for all } h \in H\},$$

is finite dimensional.

Let G be a finitely generated group and H be a subgroup of G . Let (χ, W) be a one dimensional representation of H . Let χ^g be the conjugate of χ acting on $g^{-1}Hg$. By $I(\chi, \chi^g)$, we mean the space of $H \cap g^{-1}Hg$ -linear maps from (χ, W) to (χ^g, W) when both of these are viewed as representations of the group $H \cap g^{-1}Hg$. The following result regarding the space of intertwining operators of induced representations is a

slight generalization of a result that was proved by Kutzko [12]. Its unitary analogue was proved by Mackey [13] himself in early 50's.

Theorem 2.3. *Let H be a subgroup of a discrete group G and let χ be character of H . Let Λ be the set of (H, H) -double coset representatives $g \in G$ such that HgH is union of finite number of right cosets of H in G . Then the following are true.*

- (1) *The space $\text{End}_G(\text{Ind}_H^G(\chi))$ is isomorphic to the set of functions $s : G \rightarrow \text{End}_{\mathbb{C}}(W)$ such that $s(h_1gh_2) = \chi(h_1)s(g)\chi(h_2)$ for all $h_1, h_2 \in H$, $g \in G$ and the support of s is contained in Λ .*
- (2) $\text{End}_G(\text{Ind}_H^G(\chi)) \cong \bigoplus_{g \in \Lambda} I(\chi, \chi^g)$

Proof. The proof of this follows from Kutzko [12, p.2]. Here we outline ideas of the proof. For any $w \in W$, define the set of functions $f^w : G \rightarrow W$ by

$$f^w(x) = \begin{cases} \chi(x)w & \text{for } x \in H \\ 0 & \text{otherwise.} \end{cases}$$

Then $f^w \in \text{Ind}_H^G(\chi)$. For $\psi \in \text{End}_G(\text{Ind}_H^G(\chi))$, define $s_\psi : G \rightarrow \text{End}(W)$ by $s_\psi(g)(w) = \psi f^w(g)$.

Then $\Psi : \psi \mapsto s_\psi$ gives the required isomorphism between $\text{End}_G(\text{Ind}_H^G(\chi))$ and Δ . As for the second part, for each $g \in H \backslash G / H$, let Δ_g consists of functions in Δ that have their support on HgH . Then $\Delta \cong \bigoplus_{g \in \Lambda} \Delta_g$.

For all $g \in G \backslash H / G$, we have Δ_g is isomorphic to $I(\chi, \chi^g)$ with isomorphism given by $s \mapsto s(g)$. \square

Remark 2.4. In general, when ρ is any representation of a subgroup H of G , the following result is true by *Mackey's formula* (see [18, Chapter 1, Section 5.5, 5.7], also see [4]).

$$(2.1) \quad \text{Ind}_H^G(\rho)|_H \cong \bigoplus_{g \in H \backslash G / H} \text{Ind}_{H \cap H^g}^H(\rho^g|_{H \cap H^g}).$$

This gives the following description of G -linear maps of $\text{Ind}_H^G(\rho)$.

$$(2.2) \quad \text{End}_G(\text{Ind}_H^G(\rho)) \cong \bigoplus_{g \in H \backslash G / H} \text{Hom}_H(\rho, \text{Ind}_{H^g \cap H}^H(\rho^g|_{H^g \cap H})).$$

Further in case the index of H in G is finite and π is an arbitrary representation of G , then the following vector spaces are isomorphic (see [18, Chapter 1, Section 5.4]).

$$(2.3) \quad \text{Hom}_G(\pi, \text{Ind}_H^G(\rho)) \cong \text{Hom}_H(\pi|_H, \rho).$$

If the group G is a finitely generated nilpotent group the above proposition can be simplified to give an important condition on H and $\chi : H \rightarrow \mathbb{C}^\times$ such that $\text{ind}_H^G(\chi)$ is irreducible. Recall that G is called nilpotent if every homomorphic image of G has a cyclic normal subgroup.

Definition 2.5. (Radical) A subgroup K of G is called **radical** if $x^n \in K$ for some $n \in \mathbb{N}$ implies that $x \in K$ (in [5], Brown calls K isolated).

Intersection of two radical subgroups of G is radical. Therefore given a subgroup H of G smallest radical subgroup of G containing H makes sense, we shall call this radical of H and denote this by \sqrt{H} . The following lemma lists useful properties of radical of a subgroup and their normalizers in finitely generated nilpotent groups.

Lemma 2.6. *Let H and K be subgroups of a finitely generated nilpotent group G such that $K \subseteq H$. Then the following are true.*

- (1) *If some odd power of each element of a set of generators of H lies in K then K has finite index in H and $\sqrt{H} = \sqrt{K}$.*
- (2) $\sqrt{(\sqrt{H})} = \sqrt{H}$.
- (3) $\sqrt{N_G(H)} = N_G(\sqrt{H})$.

Proof. Follows from Baumslag [3, Lemma 2.8] and Brown [5, Lemma 4]. □

Lemma 2.7. *Any $g \in G$ such that HgH is union of finite number of right cosets of H in G , has the property that for all $h \in H$ there exists $\kappa(h, g) \in \mathbb{N}$ such that $(ghg^{-1})^{\kappa(h, g)} \in H$.*

Proof. Let $g \in G$ be such that HgH is union of finite number of right cosets of H in G . Then for any $h \in H$, the sets Hgh^i can not be mutually distinct right cosets of H in G . Therefore, there exists $\kappa(h, g)$, depending on h and g , such that $Hgh^{\kappa(h, g)} = Hg$. This implies, $(ghg^{-1})^{\kappa(h, g)} \in H$ for all $h \in H$. □

Proposition 2.8. *If H is a subgroup of G , χ a character of H and for all $g \in N_G(\sqrt{H}) \setminus H$ we have $\chi^g \neq \chi$ on $H^g \cap H$, then $\text{Ind}_H^G(\chi)$ is Schur irreducible.*

Proof. By Theorem 2.3, it is enough to prove that $g \in \Lambda$ implies that $g \in N_G(\sqrt{H})$. From Lemma 2.7, we have $g \in \Lambda$ implies that $gHg^{-1} \subseteq \sqrt{H}$. Therefore $g\sqrt{H}g^{-1} \subseteq \sqrt{\sqrt{H}}$ and by Lemma 2.6(2), we get $g\sqrt{H}g^{-1} \subseteq \sqrt{H}$, that is $g \in N_G(\sqrt{H})$.

We remark that the equivalence of the one dimensionality of the space of intertwining operators and irreducibility of representations, though true for finite groups or unitary representations, need not hold for other cases. This was the gap in our earlier draft which was pointed out to us by Prof. Parshin. Below we give proof of this fact, combining ideas of [4] and [1]. □

Theorem 2.9. *Let G be a finitely generated nilpotent group. Let H be a subgroup of G and W be an irreducible representation of H . If $\text{End}_G(\text{Ind}_H^G(W)) \cong \mathbb{C}$, then the representation $\text{Ind}_H^G(W)$ is irreducible.*

Proof. First of all, we consider the case of H normal subgroup of G and W being one dimensional representation of H . We denote the action of H on W by χ and the induced representation $\text{Ind}_H^G(\chi)$ by (V, π) . By Remark 2.4, we have $V|_H = \bigoplus_{g \in G/H} W^g$. To prove that V is irreducible it is enough to show that any $v \in V$ generates V . We note that any $v \in V$ can be written as

$$(2.4) \quad v = \sum_i v_i, \quad v_i \in W^{g_i},$$

such that only finitely many are non-zero. In case, $v = v_i$ for some i then v clearly generates V because W^g is irreducible representation of H . To prove our result, we use induction on number of non-zero constituents of v . Let $v = v_1 + v_2 + \cdots + v_k$ such that $v_i \in W^{g_i}$ are all non-zero and $k \geq 2$.

The group H is normal in G implies HgH for any $g \in G$ has finite image in $H \backslash G$. Therefore by Theorem 2.3, we have $\chi^g \neq \chi$ on H for any $g \notin H$. In other words, $\chi^{g_1} \neq \chi^{g_k}$ on H for $Hg_1 \neq Hg_k$. We fix $h \in H$ such that $\chi^{g_1}(h) \neq \chi^{g_k}(h)$. Therefore we have the following,

$$(2.5) \quad ((\chi^{g_k}(h))^{-1}\pi(h) - I)(v) = \sum_{i=1}^{k-1} (\chi^{g_k}(h)^{-1}\chi^{g_i}(h) - 1)v_i \in V.$$

By the choice of h , we have $((\chi^{g_k}(h))^{-1}\pi(h) - I)(v)$ is non-trivial and has number of non-zero components strictly less than that of v . Therefore, the result follows by induction.

Next, we consider the case when group H is normal and W is of arbitrary dimension. For this case, just to make it clear, we denote action of H on W by ρ instead of χ . The only place, where the above argument is not applicable is in the definition of $((\rho^{g_k}(h))^{-1}\pi(h) - 1)$. Since $\rho(h)$ is not a scalar so we can't use this operator as such. To justify the existence of a parallel operator we use the language of group algebra and modules as given in Passman [16]. Let $\mathbb{C}[H]$ be the infinite group algebra of H over \mathbb{C} . Then for $v_1, v_k \in V$ as above, we have $\mathbb{C}[H]v_1$ and $\mathbb{C}[H]v_k$ are irreducible $\mathbb{C}[H]$ -modules. Let I^{g_1} and I^{g_k} denote the annihilators of v_1 and v_k respectively in $\mathbb{C}[H]$. Then we have the following,

$$\mathbb{C}[H]/I^{g_1} \cong \mathbb{C}[H]v_1 \cong W^{g_1} \not\cong W^{g_k} \cong \mathbb{C}[H]v_k \cong \mathbb{C}[H]/I^{g_k}.$$

The ideals I^{g_1} and I^{g_k} are clearly distinct. Therefore, there exists $\alpha \in \mathbb{C}[H]$ such that $\alpha \in (I^{g_1} \setminus I^{g_2}) \cup (I^{g_2} \setminus I^{g_1})$, $\alpha(v)$ is non-trivial and its number of non-zero components is strictly less than that of v . Now onwards, the result follows by induction in this case.

For the general case when H is a subgroup (not necessarily normal) of a finitely generated group G and W is an irreducible representation of H of an arbitrary dimension. We assume that G is nilpotent (note that this hypothesis was not required

above). Therefore, there exists a sequence of subgroups H_i for $1 \leq i \leq k$ such that,

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_k = G.$$

We call this a normal series of H of length k . We use induction on the minimum length of a normal series of a subgroup of G . In case H has a normal series of length one then result follows from above. Let result be true for all subgroups with normal series of length less than or equal to $k - 1$ and we prove it for k . For this, first we note that $\widetilde{W} = \text{Ind}_H^{H_1}(W)$ is Schur irreducible by (2.2), as $\text{End}_{H_1}(\widetilde{W})$ embeds into $\text{End}_G(\text{Ind}_H^G(W)) \cong \mathbb{C}$. The group H is normal in H_1 , therefore by above \widetilde{W} is irreducible. Next, we note that the representation $\text{Ind}_{H_1}^G(\widetilde{W}) \cong \text{Ind}_H^G(W)$ is Schur irreducible and the minimum length of a normal series of H_1 is less than k , therefore we obtain $\text{Ind}_H^G(W)$ is irreducible. \square

Lemma 2.10. *Let (π, V) be an irreducible representation of G such that $V_H(\chi) \neq 0$. Then $I(\text{Ind}_H^G(\chi), \pi) \neq 0$.*

Proof. Let ρ denote the representation $\text{Ind}_H^G(\chi)$. We prove that the space of intertwining operators $I(\text{Ind}_H^G(\chi), \pi)$ has positive dimension. Then the result follows from Schur's lemma. Let $X = \{g_i \mid i \in \mathbb{I}\}$ be the right coset representatives of H in G . For a fixed $v \in V_H(\chi)$, let W be the one dimensional space generated by v . We define functions $f_i : X \rightarrow W$ by $f_i(g_j) = \delta_{i,j}(v)$ and extend these to G so that $f_i \in \text{Ind}_H^G(\chi)$ for all $i \in \mathbb{I}$. Then any $f \in \text{Ind}_H^G(\chi)$ can be written as linear combination of f_i 's. For any $g \in G$, we have

$$(2.6) \quad \rho(g)f_i(hg_j) = f_i(hg_jg) = \chi(h)f_i(g_jgg_i^{-1}g_i)$$

Therefore $\rho(g)f_i$ is nonzero on g_j for g_j satisfying $g_jg \in Hg_i$. Thus $\rho(g)f_i = \chi(g_jgg_i^{-1})f_j$ for j such that $g_jg \in Hg_i$. Now define $F : \text{Ind}_H^G(\chi) \rightarrow \pi$ by $F(f_i) = \pi(g_i^{-1})v$ on f_i 's and extended linearly thereof. Then,

$$F(\rho(g)f_i) = F(\chi(g_jgg_i^{-1})f_j) = \chi(g_jgg_i^{-1})\pi(g_j^{-1})v = \pi(g)\pi(g_i^{-1})v.$$

This shows that $F \in I(\text{Ind}_H^G(\chi), \pi)$ is a non-zero intertwiner and therefore $\text{Ind}_H^G(\chi) \cong \pi$. \square

The above lemma implies the following useful result.

Proposition 2.11. *Let (π, V) be an irreducible representation of G such that $V_H(\chi) \neq 0$. If $\text{Ind}_H^G(\chi)$ is irreducible then $\pi \cong \text{Ind}_H^G(\chi)$. Therefore π is monomial in this case.*

The following two lemmas whose proofs are fairly standard will play a crucial role in the proof of the main result.

Lemma 2.12. *Let H and K be two subgroups of G and $\chi : H \rightarrow \mathbb{C}^\times$, $\delta : K \rightarrow \mathbb{C}^\times$ be characters of H and K respectively such that,*

- (1) $kHk^{-1} \subseteq H$ for all $k \in K$, i.e. K normalizes H .
- (2) $\chi(khk^{-1}) = \chi(h)$ for all $h \in H$ and $k \in K$.
- (3) $\chi|_{H \cap K} = \delta|_{H \cap K}$

Then $\chi\delta : HK \rightarrow \mathbb{C}^\times$ defined by $\chi\delta(hk) = \chi(h)\delta(k)$ for all $h \in H$ and $k \in K$ is a character of HK such that $\chi\delta|_H = \chi$.

Proof. The well defined-ness of $\chi\delta$ follows because $h_1k_1 = h_2k_2$ implies $h_2^{-1}h_1 = k_2k_1^{-1} \in H \cap K$. Therefore $\chi(h_2)^{-1}\chi(h_1) = \delta(k_2)\delta(k_1)^{-1}$. Rest of the proof follows easily. □

Lemma 2.13. *(see [9]) For a finitely generated group G , there exists only finitely many subgroups of a given index.*

3. PROOF OF THEOREM 1.1

In this section we prove the main theorem.

Proof. Let π be a countable dimensional irreducible representation of G having finite weight with respect to (H, χ) . We prove the existence of a subgroup H' and $\chi' : H' \rightarrow \mathbb{C}^\times$ such that the following hold

- (1) For all $g \in N_G(\sqrt{H'}) \setminus H'$ we have $(\chi')^g \neq \chi'$ on $(H')^g \cap H'$.
- (2) $V_{H'}(\chi') \neq 0$.

Then by Proposition 2.8 and Proposition 2.11, we obtain $\pi \cong \text{Ind}_{H'}^G(\chi')$ and therefore π is monomial. We divide the proof into several steps.

(a) We firstly suppose that there exists $g \in N_G(\sqrt{H}) \setminus \sqrt{H}$ such that $\chi^g = \chi$ on $H \cap H^g$.

Then, by the definition of \sqrt{H} , the element g has infinite order. Notice that, the groups H^{g^i} all have equal finite index in \sqrt{H} . Since the group \sqrt{H} is finitely generated, by Lemma 2.13, we obtain that

$$H^{g^k} = H \text{ for some } k \in \mathbb{N},$$

and

$$H_0 = \bigcap_{i=1}^{\infty} H^{g^i}.$$

has finite index in H . Therefore the infinite group generated by g^k acts on irreducible representations of H and $\chi = \chi^{g^{ki}}$ for all i on H_0 . By Lemma 2.10, we have

$$I(\text{Ind}_{H_0}^H(\chi|_{H_0}), \chi^{g^{ki}}) \neq 0 \quad \forall i \geq 0.$$

But $\text{Ind}_{H_0}^H(\chi|_{H_0})$ is finite dimensional and therefore $\chi = \chi^{g^{ki}}$ for some i . Also we have $H = H^{g^{ki}}$. Therefore there exists a character δ of group generated by g^{ki} , say K , such that

$$V_H(\chi) \cap V_K(\delta) \neq 0$$

We apply Lemma 2.12 for this H , χ , K , δ and obtain a character χ_1 of group $H_1 = \langle H, g^{ki} \rangle$ such that $H \subsetneq H_1$, and $\chi_1|_H = \chi$. It can be easily seen that $V_{H_1}(\chi_1) \neq 0$ is finite dimensional. Thus now onwards we assume that (H, χ) are chosen so that pair (H, χ) is such that torsion free rank of H is maximum with the property that $V_H(\chi) \neq 0$ is finite dimensional. From above discussion it also follows that maximal pair (H, χ) satisfies the following.

(*) The character χ of H is such that for any $g \in N_G(\sqrt{H}) \setminus \sqrt{H}$ we have $\chi^g \neq \chi$ on any finite index subgroup of H .

(b) For the next step, if for this maximal (H, χ) there exists $g \in \sqrt{H} \setminus H$ such that $\chi^g = \chi$ on $H \cap H^g$, we will modify our pair (H, χ) as follows otherwise (H, χ) itself satisfies $\pi \cong \text{Ind}_H^G(\chi)$.

Let $H_0 = \bigcap_{g \in N_G(\sqrt{H})} gHg^{-1}$, $\chi_0 = \chi|_{H_0}$ and

$$L = \{g \in G \mid \chi_0^g = \chi_0 \text{ on } H_0^g \cap H_0\}.$$

By Proposition 2.7, we have $L \subseteq N_G(\sqrt{H_0}) = N_G(\sqrt{H})$. By definition, the group H_0 is normal in $N_G(\sqrt{H})$. By (*), we get that $\chi_0^g \neq \chi_0$ for any $g \in N_G(\sqrt{H_0}) \setminus \sqrt{H_0}$. Hence it suffices to consider

$$L = \{g \in \sqrt{H} = \sqrt{H_0} \mid \chi_0^g = \chi_0\}.$$

If $H_0 = L$, then the pair (H_0, χ_0) is the required pair. If not we have H_0 is a proper normal subgroup of L . As L/H_0 is nilpotent, it has a proper cyclic normal subgroup generated by, say, gH_0 for $g \in L \setminus H_0$. Let $S = \langle g \rangle$ be the cyclic subgroup generated by g . Then H_0S is a normal subgroup of L generated by H_0 and g . Now $g \in \sqrt{H}$ gives that there exists $t \in \mathbb{N}$ such that $g^t \in H_0$. As before there exists a character $\delta : S \rightarrow \mathbb{C}^\times$ such that

$$V_{H_0}(\chi_0) \cap V_S(\delta) \neq 0$$

Then by Lemma 2.12 for these H_0 , χ_0 , S , δ , we get that there exists a character χ_1 of $H_0 \subsetneq H_1 = H_0S$ that extends χ and $V_{H_1}(\chi_1) \neq 0$.

Let $L_1 = \{g \in L \mid (\chi_1)^g = \chi_1\}$. Now

$$H_0 \triangleleft H_1 \trianglelefteq L_1 \subseteq L \subseteq \sqrt{H}.$$

If $L_1 \neq H_1$, then we obtain a group H_2 containing H_1 as a proper normal subgroup and its character χ_2 that extends χ_1 and therefore satisfying $V_{H_2}(\chi_2) \neq 0$. We note that H_0 has finite index in \sqrt{H} . Therefore continuing this way, there exists a

subgroup H_k of \sqrt{H} and a character χ_k of H_k such that $H_k = L_k$, $(\chi_k)|_{H_0} = \chi_0$, $V_{H_k}(\chi_k) \neq 0$, and $(\chi_k)^g \neq \chi_k$ on $H_k^g \cap H_k$ for any $g \in N_G(\sqrt{H_k}) \setminus H_k$. The last assertion follows by choice of H_k and (*).

Conversely, Let $\pi \cong \text{Ind}_H^G(\chi)$ be a monomial irreducible representation of G acting on representation space V . We prove that $V_H(\chi)$ is in fact a one dimensional subspace of V .

Let $\{g_i \mid i \in \mathbb{I}\}$ be a set of double coset representatives of H in G . For each $i \in \mathbb{I}$, define functions f_i such that $f_i(g_j) = \delta_{i,j}$ and then extended to the whole group G so that $f_i(hg_j) = \chi(h)f_i(g_j)$ for all j . From the definition of $\text{Ind}_H^G(\chi)$, it is clear that every $f \in V$ can be written as linear combination of f_i 's. The representation V is irreducible, therefore by Schur's lemma and Proposition 2.7 for any $g \notin H$, there exists $h \in H \cap H^g$ such that $\chi(h) \neq \chi^g(h)$. Let $f \in V$, then

$$\pi(h)f(g) = f(ghg^{-1}g) = \chi^g(h)f(g) \neq \chi(h)f(g),$$

therefore $f \in V_H(\chi)$ if and only if f is non-zero only on trivial coset representative of H in G and here it is determined by its value on identity element of H . Therefore $V_H(\chi)$ is one dimensional. \square

As a corollary, we obtain the following.

Corollary 3.1. *Every finite dimensional irreducible representation of a nilpotent group G is monomial.*

A proof of this also appears in [5, Lemma 1].

4. PROOF OF THEOREM 1.2

From the definition of $\mathbb{H}(s_1, s_2, \dots, s_n)$, it is clear that $[G, G] = \{(\mathbf{0}, \mathbf{0}, s_1 z) \mid z \in \mathbb{Z}\}$, where $\mathbf{0}$ corresponds to the zero element of abelian group \mathbb{Z}^n .

Now we prove that (1) implies (2). We have $\chi|_{[G,G]}$ is of order N_1 implies that the set $K = \{(\mathbf{0}, \mathbf{0}, N_1 s_1 z) \mid z \in \mathbb{Z}\}$ is in the kernel of ρ . Hence ρ can be considered as a representation of the group $G' = G/K$. The result that V is finite dimensional is proved if we prove existence of an abelian normal subgroup of G' of finite index.

Consider $A = \{(u, v, z) \mid u \in \oplus_{i=1}^n (N_i \mathbb{Z}), v \in \mathbb{Z}^n, z \in \mathbb{Z}\}$. Then $A' = A/K$ is a normal abelian subgroup of G' of index $N_1 N_2 \dots N_n$. Hence dimension of V is less than equal to $N_1 N_2 \dots N_n$. Let $\chi' = \chi|_{[G,G]}$ and as K is contained in the kernel of χ' we may consider χ' as a character of $[G, G]/K$. Let δ be a character of A' such that

- (1) $\delta|_{[G,G]/K} = \chi'$.
- (2) $\langle \rho|_{A'}, \delta \rangle \neq 0$.

Such a character δ exists because A' is an abelian subgroup of G' .

Lemma 4.1. *The representation $\text{Ind}_{A'}^{G'}(\delta)$ is an irreducible representation of G' .*

Proof. To prove this, we use Proposition 2.8. We consider the set

$$\{((\alpha_1, \alpha_2, \dots, \alpha_n), \mathbf{0}, 0) \mid 1 \leq \alpha_i < N_i \text{ for all } i\} \subset G$$

as the set of representatives of G'/A' . We claim that $\delta^g \neq \delta$ for all $g \in G'/A'$. This is equivalent to show that for any $g \in G'/A'$, there exists $a \in A'$ such that $\delta(gag^{-1}a^{-1}) \neq 1$.

Suppose $g = (\alpha_1, \alpha_2, \dots, \alpha_n), \mathbf{0}, 0$ is such that j is the smallest integer with $\alpha_j \neq 0$. We take $a = (\mathbf{0}, v, 0) \in G$ where $v \in \mathbb{Z}^n$ such that j th coordinate is one and all other coordinates are zero. Then $gag^{-1}a^{-1} = (\mathbf{0}, \mathbf{0}, \alpha_j s_j)$. But then by the choice, $\alpha_j s_j < N_j s_j$ and therefore $\alpha_j s_j$ is not multiple of N_1 , this implies that $\delta(gag^{-1}a^{-1}) = \chi_{[G, G]}(gag^{-1}a^{-1}) \neq 1$.

□

Therefore $\text{Ind}_{A'}^{G'}(\delta)$ is irreducible. Hence by Lemma 2.11, we obtain that $\rho \cong \text{Ind}_{A'}^{G'}(\delta)$. Therefore dimension of ρ is equal to $N_1 N_2 \dots N_n$.

To prove that (2) implies (1), let $\dim(V) = N_1 N_2 \dots N_n$ such that N_i are defined as above. Since $\dim(V) < \infty$, ρ is monomial. Hence each $\rho(g)$ is a monomial matrix and it follows that there exists a large positive integer m such that $\rho(g)^m$ is diagonal for all $g \in G$. Let $a, b \in G$ be such that $[a, b] = (\mathbf{0}, \mathbf{0}, s_1) \in [G, G]$. Since $[a^m, b^m] = [a, b]^{m^2}$ (see proof of Lemma 5.1 below), we have

$$I = \rho[a^m, b^m] = \chi[a^m, b^m] = \chi(\mathbf{0}, \mathbf{0}, m^2 s_1).$$

Hence χ has order less than or equal to m^2 .

We prove that order of χ is N_1 . Suppose order of χ is d_1 . But then by the first part, we have $\dim(V) = d_1 d_2 \dots d_n$. Here, definition of d_j are similar to that of N_j in the statement of theorem. Since $N_j a_j = b_j N_1$, where a_j, b_j are positive integers, it follows from the definition of N_j that N_j and b_j do not have common factors and so N_j divide N_1 . Similarly, d_j divide d_1 . Notice that $\dim(V) = N_1 N_2 \dots N_n = d_1 d_2 \dots d_n$. So the prime factors appearing in N_1 and d_1 will have to be the same. Let $N_1 = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ and $d_1 = p_1^{f_1} p_2^{f_2} \dots p_k^{f_k}$ with $p_i < p_{i+1}$. Suppose j is smallest integer such that $e_j \neq f_j$. Let $e_j < f_j$. Then by definition of d_i and N_i , we have the power of p_j appearing in each d_i is less than or equal to that of N_j with strict inequality for d_1 and N_1 . But then this contradicts the fact that $\dim(V) = N_1 N_2 \dots N_n = d_1 d_2 \dots d_n$.

Closely following the proof above we obtain the following corollary, analogues to Theorem 4 in [2].

Corollary 4.2. *Let $G = \mathbb{H}(1, 1, \dots, 1)$, and let $V = \text{Ind}_{H(V)}^G(\chi)$ be an irreducible representation of G where $\chi : H(V) \rightarrow \mathbb{C}^\times$ is a character. Let χ_C be the restriction of χ to the center $\{(\mathbf{0}, \mathbf{0}, z) : z \in \mathbb{Z}\}$. Then the following are equivalent.*

- (1) V is finite dimensional and $\dim V = N^n$.
- (2) χ_C has order $N < \infty$ in \mathbb{C}^\times .
- (3) $[G : H(V)] = N^n < \infty$.
- (4) $H(V) = (N\mathbb{Z})^n \times \mathbb{Z}^n \times \mathbb{Z}$.

5. PROOF OF THEOREM 1.3

In this section, we generalize some of the results of the last section to all finitely generated two step nilpotent groups.

Lemma 5.1. *Let G be a finitely generated, two step nilpotent group such that $[G, G]$ is finite cyclic. Then $G/Z(G)$ is finite.*

Proof. Let k be the order of $[G, G]$ we show that $x^k \in Z(G)$ for all $x \in G$. First notice that $[x^\ell, y] = [x, y]^\ell$. This is easily proved by induction. For,

$$[x^{\ell+1}, y] = x^{\ell+1} y x^{-\ell-1} y^{-1} = x[x^\ell, y] y x^{-1} y^{-1} = [x^\ell, y][x, y].$$

It follows that $[x^k, y] = e$ for all $y \in G$, and so $x^k \in Z(G)$. Since $G/Z(G)$ is a finitely generated abelian group and every element in $G/Z(G)$ has finite order it follows that $G/Z(G)$ is finite. □

Theorem 5.2. *Let G be a finitely generated two step nilpotent group ρ be an irreducible finite dimensional representation (and so monomial) of G . Let $G_\rho = G/\text{Ker } \rho$, then $\dim \rho = \sqrt{|G_\rho/Z(G_\rho)|}$.*

Proof of this theorem requires some lemmas.

Lemma 5.3. *Let \mathcal{G} be a finite abelian group and $f : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C}^\times$ be an anti-symmetric non-degenerate co-cycle ($f \in H^2(\mathcal{G}, \mathbb{C}^\times)$). Then $\mathcal{G} = A \times \hat{A}$ where A is finite abelian and \hat{A} is the group of characters of A . Moreover, any maximal abelian normal subgroup of \mathcal{G} has index $|A|$ in \mathcal{G} .*

For a proof see [6, Lemma 4.2].

Before we state the next lemma we introduce some notation. Let \mathbb{Z}_p be the cyclic group of order p . By \mathbb{Z}_p^2 we denote the direct product $\mathbb{Z}_p \times \mathbb{Z}_p$. Also recall that the character group of \mathbb{Z}_p is isomorphic to itself.

Lemma 5.4. *Under the assumptions in Theorem 5.2*

$$G_\rho/Z(G_\rho) \cong \mathbb{Z}_{p_1}^2 \times \mathbb{Z}_{p_2}^2 \times \dots \times \mathbb{Z}_{p_k}^2.$$

Proof. Let χ be the character obtained by restricting ρ to the center $Z(G_\rho)$. Since ρ is faithful, so is χ . Define

$$f : G_\rho/Z(G_\rho) \times G_\rho/Z(G_\rho) \rightarrow \mathbb{C}^\times$$

by

$$(5.1) \quad f(xZ(G_\rho), yZ(G_\rho)) = \chi[x, y].$$

It is easily seen that this is well defined. Since $f(y, x) = \chi[y, x] = \chi[x, y]^{-1} = f(x, y)^{-1}$ it follows that f is anti-symmetric. If x is not in $Z(G)$ there exists y such that $xy \neq yx$. Hence $[x, y] \neq 1$, and since χ is faithful, it follows that f is non-degenerate. Next, we show that $[G_\rho, G_\rho]$ is finite cyclic. Since ρ is monomial, each $\rho(g)$ is a monomial matrix. If $a, b \in G_\rho$, $aba^{-1}b^{-1} \in [G_\rho, G_\rho] \subset Z(G_\rho)$. Let m be a large enough integer such that $\rho(g)^m$ is diagonal for all $g \in G_\rho$. It follows that $I = \rho[a^m, b^m] = \chi[a^m, b^m]$. Hence $\chi|_{[G_\rho, G_\rho]}$ has order less than or equal to $m^2 < \infty$. Since $\chi|_{[G_\rho, G_\rho]}$ is faithful, it follows that $[G_\rho, G_\rho]$ is finite cyclic. Applying Lemma 5.1 we have $G_\rho/Z(G_\rho)$ is finite and is abelian as $G_\rho/Z(G_\rho) \subset G_\rho/[G_\rho, G_\rho]$. By Lemma 5.3 we have $G_\rho/Z(G_\rho) \cong A \times \hat{A}$ for some A and from the structure theorem for abelian groups we have the proof. \square

Now we are in a position to complete the proof of Theorem 5.2. Choose H_ρ so that $Z(G_\rho) \subset H_\rho$ and H_ρ is maximal with respect to the property

$$(5.2) \quad \forall (x, y) \in H_\rho \times H_\rho \quad f(xZ(G_\rho), yZ(G_\rho)) = 1$$

where f is given by 5.1.

For $x, y \in H_\rho$, $f(x, y) = 1 \implies [H_\rho, H_\rho] \subset \text{Ker} \chi = e$. It follows that H_ρ is abelian and χ extends as a character of H_ρ , say $\tilde{\chi}$. Moreover, since $Z(G_\rho) \subset H_\rho$ we have that H_ρ is normal in G_ρ . If H is any normal abelian group such that $H_\rho \subset H$, then H will be equal to H_ρ due to maximality of H_ρ . From Lemma 5.3 we have that $|G_\rho/H_\rho| = p_1 p_2 \dots p_k$. Next, we claim that the stabilizer of $\tilde{\chi}$ in G_ρ equals H_ρ . Now, if there exists $g \in G_\rho$ such that $\tilde{\chi}^g(x) = \tilde{\chi}(x) \forall x \in H_\rho$ we obtain that $\tilde{\chi}([g, x]) = 1$ for all $x \in H_\rho$. If g is not in H_ρ this will contradict the maximality. By Proposition 2.8 $\text{Ind}_{H_\rho}^{G_\rho}(\tilde{\chi})$ is irreducible and by Proposition 2.11 it is equivalent to ρ . This completes the proof of Theorem 5.2.

Conversely, we have

Theorem 5.5. *Let G be a finitely generated two step nilpotent group and ρ an irreducible representation of G . Assume that the character χ obtained by restricting ρ to $[G, G]$ has finite order in \mathbb{C}^\times . Then ρ is finite dimensional and $\dim \rho = \sqrt{|G_\rho/Z(G_\rho)|}$ where G_ρ is $G/\text{Ker} \rho$.*

Proof. Since ρ is faithful on G_ρ it produces a faithful character χ (we use the same notation) when restricted to $[G_\rho, G_\rho]$. Since χ has finite order, it follows that $[G_\rho, G_\rho]$ is finite cyclic. Now, we may proceed as above. \square

Now to prove that order of χ divides dimension of ρ , It suffices to consider $G_\rho = G/\text{Ker}\rho$. Using the previous notation, we show that $\chi^{p_1 p_2 \dots p_k} = 1$. Note that, if $x \in G_\rho$, then $x^{p_1 p_2 \dots p_k} \in Z(G_\rho)$. Hence, if $x, y \in G_\rho$,

$$\chi^{p_1 p_2 \dots p_k}[x, y] = \chi[x, y]^{p_1 p_2 \dots p_k} = \chi[x^{p_1 p_2 \dots p_k}, y] = 1.$$

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